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A Note on Simultaneous Rootfinding for Algebraic, Exponential, and Trigonometric Polynomials

C. CARSTENSEN

Institut für Angewandte Mathematik, Universität Hannover

Welfengarten 1, D-30167 Hannover, Germany

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Abstract—A common derivation and convergence analysis is presented for two methods for the simultaneous computation of all the zeros of an algebraic, exponential, or trigonometric polynomial. The analysis is performed considering a certain class of generalized polynomials which allow a factorization in terms of translations of a function $q(t)$ such that Lagrangian-like interpolation is exact. The methods were already known for algebraic polynomials ($q(t) = t$) but new for exponential ($q(t) = \sinh(t/2)$) and trigonometric polynomials ($q(t) = \sin(t/2)$). Using the properties $q(0) = 0$, $q'(0) \neq 0$, and $q''(0) = 0$, we prove the local convergence of the methods presented in this note. The algorithms only require the evaluation of the generalized polynomial at certain points but no derivatives or coefficients. Numerical examples are included.

Keywords—Iteration algorithm, Simultaneous rootfinding, Generalized polynomials.

1. INTRODUCTION

Whereas the simultaneous computation of all the zeros of an algebraic polynomial is mathematically well understood (cf., e.g., [1]), comparably little is known about related methods for generalized polynomials like trigonometric or exponential polynomials. The “homotopy technique” was extended for simultaneous rootfinding of generalized polynomials in [2,3]. Frommer [4] presented a unified approach for (known) methods with local quadratic convergence which was extended to higher order methods in [5].

This note deals with a class of generalized polynomials, namely the class of all functions $f(t)$ with n simple zeros ζ_1, \dots, ζ_n , such that there exists a function $q(t)$ satisfying first

$$f(t) = c^* \cdot \prod_{k=1}^n q(t - \zeta_k), \quad (1)$$

for a constant $c^* \neq 0$, and for all real or complex arguments t (at least in a neighbourhood of ζ_j), and secondly that for sufficient good approximants x_1, \dots, x_n of ζ_1, \dots, ζ_n there exist constants c_0, \dots, c_n with

$$f(t) = \sum_{j=0}^n c_j \cdot \prod_{k=0, k \neq j}^n q(t - x_k). \quad (2)$$

It is shown, in Section 2, that algebraic, exponential, and trigonometric polynomials belong to the class of generalized polynomials for

$$q(t) = t, \quad q(t) = \sinh\left(\frac{t}{2}\right), \quad \text{and} \quad q(t) = \sin\left(\frac{t}{2}\right), \quad (3)$$

respectively.

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The aim of this note is to study the general situation and to present generalizations of the third-order method of Maehly, Aberth and Ehrlich and a fourth-order modification of Nourain to methods for simultaneous rootfinding for exponential and trigonometric polynomials which do not use derivatives. We refer to [1] for a description of the above-mentioned methods in the algebraic case and for related references.

In this context, it should be mentioned that Weidner [6] proposed the transformation of exponential and trigonometric polynomials into algebraic polynomials with a subsequent computation of all the zeros with, e.g., Weierstrass' method. Since the new methods in this paper are not faster than corresponding methods for algebraic polynomials, it seems that Weidner's transformation will be preferably used in practice. But the methods of this note are interesting since, first, only real arithmetic is required when all zeros as well as all coefficients are real (while Weidner's transformation always needs complex arithmetic for trigonometric polynomials). Secondly, if $f(t)$ is given in a form which directly allows a pointwise evaluation of $f(t)$ but does not give all its coefficients a_0, \dots, a_n and b_1, \dots, b_n of the "standard" representation, e.g., in the trigonometric case

$$\sum_{j=0}^n a_j \cdot \cos(j \cdot t) + \sum_{j=1}^n b_j \cdot \sin(j \cdot t),$$

the coefficients do not have to be computed (while Weidner's transformation needs further effort to compute those coefficients which can be both less efficient and more instable than a direct computation).

After some preliminaries in Section 2, we present generalizations of the method of Durand-Kerner, Maehly-Aberth-Ehrlich and Nourain in Section 3, Section 4 and Section 5, respectively. Our approach is based on Newton-Jacobi and *regula falsi*-Jacobi methods. A numerical example is given in Section 6 where it turns out that the new fourth-order method is the most efficient of the known direct methods for exponential and trigonometric polynomials.

2. PRELIMINARIES

Let $f : G \rightarrow \mathbb{K}$ be an analytical mapping having exactly n simple zeros ζ_1, \dots, ζ_n in the open set G of \mathbb{K} , where \mathbb{K} denotes the real or complex field, respectively. Let $q : \mathbb{K} \rightarrow \mathbb{K}$ be smooth with $q(0) = 0 \neq q'(0)$ and let there exist $\alpha_3, \alpha_4, \dots \in \mathbb{K}$ such that

$$q(t) = q'(0) \cdot \{t + \alpha_3 t^3 + \alpha_4 t^4 + \dots\}. \quad (4)$$

Here, in view of our examples, we assume that $\alpha_2 = 0 = q''(0)$ in the sequel (i.e., $\alpha_2 t^2$ is missing in (4)). For any function $q(t)$ as in (4), we can find open subsets U_1, \dots, U_n of G with $\zeta_j \in U_j \subseteq G$ and a fixed $x_0 \in U^c := G \setminus (U_1 \cup \dots \cup U_n)$ such that for all $(x_1, \dots, x_n) \in U := U_1 \times \dots \times U_n$, there holds $q(x_j - x_k) \neq 0$ if $j \neq k$, $j, k = 0, 1, \dots, n$.

Throughout this note, we assume that there holds (1) and that for any $(x_1, \dots, x_n) \in U$ and $x_0 \in U^c$, we have c_0, \dots, c_n with (2) for any $t \in G$.

REMARK 1. For any example, we have a factorization (1) of $f(t)$ with

$$c^* := \frac{f(x_0)}{\prod_{k=1}^n q(x_0 - \zeta_k)} \neq 0. \quad (5)$$

This is also used in [2,3]. The coefficients in (2) are given by

$$c_j := \frac{f(x_j)}{\prod_{k=0, k \neq j}^n q(x_j - x_k)}, \quad (j = 1, \dots, n). \quad (6)$$

EXAMPLE 1. If $f(t)$ is an algebraic polynomial of (exact) degree n with coefficients in \mathbb{K} , then $q(t) = t$ satisfies the above assumptions. Note that in the algebraic case, we can choose $x_0 = \infty$. Then $c_0 = c^*$ is the leading coefficient of $f(t)$ and the terms like $q(x_0 - x_k)$, $k = 1, \dots, n$, can be replaced by c^* in (5) and (6).

EXAMPLE 2. If $f(t)$ is an exponential polynomial of (exact) degree $n/2$ (n is even) with coefficients in \mathbb{K} , then $q(t) = \sinh(t/2)$ satisfies the above assumptions.

EXAMPLE 3. If $f(t)$ is a trigonometric polynomial of (exact) degree $n/2$ (n is even) with coefficients in \mathbb{K} , then $q(t) = \sin(t/2)$ satisfies the above assumptions.

REMARK 2. The statements of the three examples are e.g., proved as follows: Using simple calculations, it is seen that the right hand sides of (1) and (2) are indeed polynomials of degree n . According to (5) and (6) the identities (1) and (2) hold for $t = x_0, \zeta_1, \dots, \zeta_n$ and $t = x_0, \dots, x_n$, respectively, and consequently for all t .

The new methods in this note are based on the following functions $h_1, \dots, h_n : G \rightarrow \mathbb{K}$, defined for $(x_1, \dots, x_n) \in U$ through

$$h_j(t) := \frac{f(t)}{\prod_{k=1, k \neq j}^n q(t - x_k)}, \quad (j = 1, \dots, n). \quad (7)$$

We choose x_1, \dots, x_n as approximants for the zeros ζ_1, \dots, ζ_n of $f(t)$ and are interested in the zeros of $h_j(t)$. Therefore, we apply successive iteration, Newton-Raphson's method or the *regula falsi* to $h_j(t)$ in the sequel. Note that this is just a Jacobi method for $F : U \rightarrow \mathbb{K}^n$ with $F(x) = (h_1(x_1), \dots, h_n(x_n))$.

It will follow from Lemma 1 below that the choice of $h_j(t)$ yields an improvement of convergence. Indeed, consider an iterative method for the computation of zeros of $h_j(t)$ with convergence order k such that the asymptotic constant of the iterative method is given by a derivative of $h_j(t)$. Then Lemma 1 shows that the simultaneous application of the iterative method yields convergence order $k + 1$.

LEMMA 1. For $j = 1, \dots, n$, there holds

$$\begin{aligned} \frac{h_j(\zeta_j + \tau)}{c^* q'(0)} &= (1 + \beta_0)\tau + (\beta_1 + \alpha_2(1 + \beta_0))\tau^2 \\ &\quad + (\beta_2 + \alpha_2\beta_1 + \alpha_3(1 + \beta_0))\tau^3 \\ &\quad + (\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_4(1 + \beta_0))\tau^4 + \dots, \end{aligned} \quad (8)$$

where, using Landau's symbol, for any $\nu \geq 0$

$$\beta_\nu = O\left(\max_{k=1, \dots, n, k \neq j} |x_k - \zeta_k|\right). \quad (9)$$

PROOF. By (2),

$$h_j(\zeta_j + \tau) = c^* q(\tau) \cdot (1 + r_j(\zeta_j + \tau)),$$

where

$$1 + r_j(\zeta_j + \tau) = \prod_{k=1, k \neq j}^n \frac{q(\tau + \zeta_j - \zeta_k)}{q(\tau + \zeta_j - x_k)} =: 1 + \beta_0 + \beta_1\tau + \beta_2\tau^2 + \dots.$$

r_j depends on x_1, \dots, x_n in a way that any coefficient satisfies (9). Multiplying the power series of q and $1 + r_j$ proves (8). ■

3. A SECOND-ORDER METHOD

In this section, we apply successive iteration to $h_j(t)$ which leads to a known second-order method. This section is added not only for completeness but also to stress the Weierstrass' corrections which will be used in the new fourth-order method in Section 5.

With given approximants $(x_1, \dots, x_n) \in U$ and a fixed x_0 one step of the considered method reads

$$\hat{x}_j := x_j - \omega h_j(x_j), \quad (j = 1, \dots, n), \quad (10)$$

where $\omega \in \mathbb{K} \setminus \{0\}$ is determined below.

Define $\epsilon_j := x_j - \zeta_j$, $\hat{\epsilon}_j := \hat{x}_j - \zeta_j$. Lemma 1 and Equation (10) give

$$\hat{\epsilon}_j = \epsilon_j(1 - \omega c^* q'(0)(1 + \beta_0)) - \omega c^* q'(0) \epsilon_j^2 (\beta_1 + \alpha_2(1 + \beta_0)) + \dots \quad (11)$$

so ω should be chosen such that $\omega \approx 1/(c^* q'(0))$. A natural approximation of c^* is achieved by replacing the (unknown) zeros in (5) by their (known) approximants x_1, \dots, x_n which leads to the choice

$$\omega = \frac{\prod_{k=1}^n q(x_0 - x_k)}{f(x_0)q'(0)} = \frac{1}{c_0 q'(0)}. \quad (12)$$

Since then, $\omega q'(0) = c^*(1 + O(\epsilon))$, $\epsilon := \max_{k=1, \dots, n} |\epsilon_k|$, method (10) with ω given in (12), i.e.,

$$\hat{x}_j = x_j - W_j, \quad W_j := \frac{h_j(x_j)}{c_0 q'(0)} = \frac{c_j q(x_j - x_0)}{c_0 q'(0)}, \quad (j = 1, \dots, n), \quad (13)$$

is of second-order.

Method (13) is known from [2,3] and has various derivations, see [1,4,5], and the references therein.

Note that for the algebraic case with $x_0 = \infty$, the approximation of c^* is not necessary and $1/\omega$ is the leading coefficient of $f(t)$. Then (13) is the famous Durand-Kerner's method going back to Weierstrass (see, e.g., [1]). Therefore, W_j is called a Weierstrass' correction. With $\alpha_\nu = 0$ for $\nu \geq 2$ and Lemma 1, we obtain the following error estimate from (11)

$$\hat{\epsilon}_j = O\left(\epsilon_j \cdot \max_{k=1, \dots, n, k \neq j} |x_k - \zeta_k|\right). \quad (14)$$

In the general case (3) and $x_0 \neq \infty$, (14) is false and must be replaced by

$$\hat{\epsilon}_j = O\left(\epsilon_j \cdot \max_{k=1, \dots, n} |x_k - \zeta_k|\right). \quad (15)$$

REMARK 3. Note that in (14), $\hat{\epsilon}_j = 0$ if $x_k = \zeta_k$ for all $k \neq j$ but, in general, not in (15). This property has the effect that the so-called single step mode converges faster than the original method (10) which is called total step mode. Consequently, the faster convergence of single step methods cannot be proved in the general case with the technique working in the algebraic case with $x_0 = \infty$. For a description of the single step modes and the corresponding convergence proofs we refer to [1] and, in particular, to Theorem 2.4 therein.

We will see below that this is a typical disadvantage for the other methods of this note in the case $x_0 \neq \infty$.

4. A THIRD-ORDER METHOD

Using the notation from the previous sections, we apply Newton-Raphson's method to $h_j(t)$, i.e., one step of our method reads:

$$\hat{x}_j := x_j - \frac{h_j(x_j)}{h'_j(x_j)}, \quad (j = 1, \dots, n). \quad (16)$$

The following theorem proves that (16) is a generalization of the third-order method of Maehly, Aberth, Ehrlich.

THEOREM 1. *Method (16) is locally convergent of order three with the alternative representations*

$$\hat{x}_j = x_j - \frac{1}{N(x_j)^{-1} - \sum_{k=1, k \neq j}^n \frac{q'(x_j - x_k)}{q(x_j - x_k)}}, \quad (j = 1, \dots, n)$$

and

$$\hat{x}_j = x_j - \frac{1}{\frac{q'(x_j - x_0)}{q(x_j - x_0)} + \frac{q'(0)}{c_j} \sum_{k=0, k \neq j}^n \frac{c_k}{q(x_j - x_k)}}, \quad (j = 1, \dots, n),$$

where $N(x_j) := f(x_j)/f'(x_j)$ is Newton's correction and c_0, \dots, c_n are given in (6).

PROOF. Some calculations using (7), (2), and (4) lead to the required representations of \hat{x}_j . To determine the local convergence properties, we apply Lemma 1 and finally obtain (writing $\epsilon_j := x_j - \zeta_j$, $\hat{\epsilon}_j := \hat{x}_j - \zeta_j$)

$$\hat{\epsilon}_j = \frac{\epsilon_j^2(\alpha_2(1 + \beta_0) + \beta_1) + 2\epsilon_j^3(\alpha_3(1 + \beta_0) + \alpha_2\beta_1 + \beta_2) + \dots}{1 + o(1)}.$$

Here, Landau's symbol $o(1)$ is defined by $\lim_{(x_1, \dots, x_n) \rightarrow (\zeta_1, \dots, \zeta_n)} o(1) = 0$.

Note that $\alpha_2 = 0$ implies

$$\hat{\epsilon}_j = O\left(\epsilon_j^2 \cdot \max_{k=1, \dots, n} |x_k - \zeta_k|\right), \quad (17)$$

such that (16) is a third-order method for the general case (3). ■

We finally discuss possible modifications of (16). A numerical example is given in the last section.

REMARK 4. Note that for the algebraic case with $x_0 = \infty$, we obtain the improved error estimate

$$\hat{\epsilon}_j = O\left(\epsilon_j^2 \cdot \max_{k=1, \dots, n, k \neq j} |x_k - \zeta_k|\right) \quad (18)$$

giving faster convergence for the single step mode. As mentioned in Remark 4, this is false if $x_0 \neq \infty$. Note also that (18) would allow the use of further corrections replacing x_k by $x_k - N(x_k)$ in the first representation in Theorem 1. This defines the new method

$$\hat{x}_j = x_j - \frac{1}{N(x_j)^{-1} - \sum_{k=1, k \neq j}^n \frac{q'(x_j - x_k - N(x_k))}{q(x_j - x_k - N(x_k))}}, \quad (j = 1, \dots, n). \quad (19)$$

According to the quadratic convergence of Newton-Raphson's method, (18) yields the following error estimate for the fourth-order method (19)

$$\hat{\epsilon}_j = O\left(\epsilon_j^2 \cdot \max_{k=1, \dots, n, k \neq j} |x_k - \zeta_k|^2\right).$$

Unfortunately, because of (17), method (19) is only of third-order for the general case (3).

5. A FOURTH-ORDER METHOD

Using the notation from the previous sections, we apply *regula falsi* to $h_j(t)$, i.e., one step of our method reads

$$\hat{x}_j := x_j - \frac{\tilde{x}_j - x_j}{h(\tilde{x}_j) - h(x_j)} h(x_j), \quad (j = 1, \dots, n), \quad (20)$$

where \tilde{x}_j is a second approximant for ζ_j which may depend on x_j . For instance, Newton's $\tilde{x}_j = x_j - N(x_j)$ or Weierstrass' correction $\tilde{x}_j = x_j - W_j$, W_j given in (13), may be applied.

Define $\epsilon_j := x_j - \zeta_j$, $\hat{\epsilon}_j := \hat{x}_j - \zeta_j$, $\tilde{\epsilon}_j := \tilde{x}_j - \zeta_j$ to describe the convergence properties of method (20).

THEOREM 2. *For method (20), there holds*

$$\hat{\epsilon}_j = O\left(\epsilon_j \cdot \tilde{\epsilon}_j \cdot \max_{k=1, \dots, n} \{|\epsilon_k| + |\tilde{\epsilon}_j|\}\right). \quad (21)$$

PROOF. According to (8), it follows

$$\frac{\epsilon_j h(\tilde{x}_j) - \tilde{\epsilon}_j h(x_j)}{c^* q'(0)(\tilde{\epsilon}_j - \epsilon_j)} = \epsilon_j \tilde{\epsilon}_j \left((\beta_1 + \alpha_3(\tilde{\epsilon}_j + \epsilon_j)) + \dots \right) = O\left(\epsilon_j \cdot \tilde{\epsilon}_j \cdot \max_{k=1, \dots, n} \{|\epsilon_k| + |\tilde{\epsilon}_j|\}\right).$$

Since

$$\begin{aligned} \frac{h(\tilde{x}_j) - h(x_j)}{\tilde{\epsilon}_j - \epsilon_j} &= \frac{h(\tilde{x}_j) - h(x_j)}{\tilde{x}_j - x_j} \rightarrow h'(\zeta_j) \neq 0, \quad \text{if } \tilde{x}_j, x_j \rightarrow \zeta_j, \\ \hat{\epsilon}_j &:= \frac{\epsilon_j h(\tilde{x}_j) - \tilde{\epsilon}_j h(x_j)}{\tilde{\epsilon}_j - \epsilon_j} \cdot \frac{h(\tilde{x}_j) - h(x_j)}{\tilde{\epsilon}_j - \epsilon_j} = O\left(\epsilon_j \cdot \tilde{\epsilon}_j \cdot \max_{k=1, \dots, n} \{|\epsilon_k| + |\tilde{\epsilon}_j|\}\right). \quad \blacksquare \end{aligned}$$

THEOREM 3. *For method (20), the following representation holds (c_0, \dots, c_n are given in (6))*

$$\hat{x}_j = x_j + \frac{\tilde{x}_j - x_j}{1 - \frac{q(\tilde{x}_j - x_0)}{q(x_j - x_0)} \frac{q(\tilde{x}_j - x_j)}{c_j} \sum_{k=0}^n \frac{c_k}{q(\tilde{x}_j - x_k)}}, \quad (j = 1, \dots, n). \quad (22)$$

Using Weierstrass' corrections $\tilde{x}_j := x_j - W_j$ we get

$$\hat{x}_j = x_j - \frac{W_j}{1 - \frac{q(x_j - W_j - x_0)q(-W_j)}{c_0 q'(0)W_j} \sum_{k=0}^n \frac{c_k}{q(x_j - W_j - x_k)}}, \quad (j = 1, \dots, n). \quad (23)$$

PROOF. Apply (2) and (1) to obtain

$$h_j(\tilde{x}_j) = q(\tilde{x}_j - x_0) q(\tilde{x}_j - x_j) \sum_{k=0}^n \frac{c_k}{q(\tilde{x}_j - x_k)},$$

from which the proof of the theorem can be derived by some calculations. ■

REMARK 5. Note that the fourth-order method (23) is very efficient. To see this, note that after the computation of W_j the constants c_0, \dots, c_n are known as well (cf. (13)). Thus, to improve the convergence from second- to fourth-order via (23) requires (essentially) only further evaluation of $q(x_j - W_j - x_k)$, a division and addition for $k = 0, \dots, n$ and $j = 1, \dots, n$. Consequently, the computational costs of the fourth- and third-order method (23) and (16) are of the same complexity.

REMARK 6. In the algebraic case with $x_0 = \infty$, method (20) is a modification due to Nourain (cf. [1,7]). Then, since $\alpha_\nu = 0$ for $\nu \geq 2$, (21) can be improved to

$$\hat{\epsilon}_j = O\left(\epsilon_j^2 \max_{k=1, \dots, n, k \neq j} |\epsilon_k|^2\right),$$

and, as mentioned in Remark (3), the related single step mode converges faster than the total step mode.

REMARK 7. Any correction $\tilde{\zeta}_j$ of order k , i.e., $\tilde{\epsilon}_j = (\epsilon_j^k)$, in method (22) yields the convergence order $(k + 2)$.

6. NUMERICAL EXAMPLE

We present only one example for the exponential case. Let $f(t) = \prod_{j=1}^4 \sinh((t - \zeta_j)/2)$ with $\zeta_1 = -1$, $\zeta_2 = 2$, $\zeta_3 = 3$, $\zeta_4 = 4$. The example is taken from [3] where also a third-order method (cf. (2.13) therein))

$$x_i^{(\nu+1)} := x_i^{(\nu)} - 4C_\nu f(x_i^{(\nu)}) \frac{V_i^{(\nu)} - C_\nu f'(x_i^{(\nu)}) + C_\nu f(x_i^{(\nu)}) W_i^{(\nu)}}{V_i^{(\nu)^2}}, \quad (24)$$

for $i = 1, \dots, n$, $\nu = 0, 1, 2, \dots$ is considered with the notation

$$V_i^{(\nu)} := \prod_{j=1, j \neq i}^n \sinh\left(\frac{x_i^{(\nu)} - x_j^{(\nu)}}{2}\right), \quad W_i^{(\nu)} := \frac{1}{2} \sum_{j=1, j \neq i}^n \coth\left(\frac{x_i^{(\nu)} - x_j^{(\nu)}}{2}\right)$$

and $C_\nu := 1/c_0$ with c_0 defined in (6). x_0 is an arbitrary number with $f(x_0) \neq 0$; we chose $x_0 = 0$.

The following Tables 1, 2, and 3 show the errors $\epsilon_j^{(\nu)}$ of the approximants $x_j^{(\nu)}$ obtained by the methods (24), (16), and (23), respectively, using the initial values $x_1^{(0)} = -0.5$, $x_2^{(0)} = 1.7$, $x_3^{(0)} = 2.6$, $x_4^{(0)} = 4.3$. The calculations are carried out on a personal computer in 18 digits floating point arithmetic.

Table 1. Errors for method (24).

ν	$\epsilon_1^{(\nu)}$	$\epsilon_2^{(\nu)}$	$\epsilon_3^{(\nu)}$	$\epsilon_4^{(\nu)}$
0	5.0000E - 0001	-3.0000E - 0001	-4.0000E - 0001	3.0000E - 0001
1	1.4348E - 0001	-3.5850E - 0002	-2.2059E - 0001	1.5298E - 0001
3	3.9540E - 0003	2.6875E - 0004	-1.9313E - 0002	1.6831E - 0002
4	9.6390E - 0008	1.1401E - 0008	-1.4644E - 0005	1.4601E - 0005
5	2.1684E - 0019	4.3368E - 0019	-7.0610E - 0015	7.0608E - 0015
6	2.7105E - 0019	-2.1684E - 0019	-4.3368E - 0019	0.0000E + 0000

Table 2. Errors for method (16).

ν	$\epsilon_1^{(\nu)}$	$\epsilon_2^{(\nu)}$	$\epsilon_3^{(\nu)}$	$\epsilon_4^{(\nu)}$
0	5.0000E - 0001	-3.0000E - 0001	-4.0000E - 0001	3.0000E - 0001
1	2.1481E - 0002	1.9329E - 0002	7.0617E - 0002	1.7301E - 0002
2	-3.3814E - 0007	-2.4555E - 0005	-1.4207E - 0004	-2.0942E - 0005
3	6.5052E - 0019	7.9932E - 0014	6.0661E - 0013	5.8542E - 0014
4	2.1684E - 0019	-2.1684E - 0019	-4.3368E - 0019	0.0000E + 0000

Table 3. Errors for method (23).

ν	$\epsilon_1^{(\nu)}$	$\epsilon_2^{(\nu)}$	$\epsilon_3^{(\nu)}$	$\epsilon_4^{(\nu)}$
0	5.0000E - 0001	-3.0000E - 0001	-4.0000E - 0001	3.0000E - 0001
1	9.4548E - 0003	1.0254E - 0002	4.0730E - 0002	1.2919E - 0002
2	-7.0013E - 0010	-8.6099E - 0008	9.0317E - 0007	5.6888E - 0007
3	1.6263E - 0019	-1.0842E - 0019	-4.3368E - 0019	4.3368E - 0019

From Table 1,2,3 the theoretical convergence orders are confirmed.

We report that, in this example, method (13) needs 7 steps and Newton-Raphson's method $(x_j^{(\nu+1)} := x_j^{(\nu)} - f(x_j^{(\nu)})/f'(x_j^{(\nu)}))$ needs 11 steps to compute a result with an error 10^{-19} .

Some experiences showed that (16) is slightly better than (24) but requires slightly less computational effort and avoids the norming constant C_0 and, in particular, the calculation of a number

x_0 with $f(x_0) \neq 0$. Changing the initial values, it can be observed that the convergence domain of (16) seems to be greater than the convergence domain of (24) or (13).

The computational effort of the three methods considered is of the same complexity albeit (23) is of fourth-order.

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